Exam 1

Math 253, Spring 2021

Name: _____

Tell me what you know: If you don't know the entire answer, showing a formula or writing something illustrating that you understand any concept involved in the problem will allow me to give partial credit. I have to give you a 0 if you write nothing down. You will earn more points by writing down one true thing and no nonsense things than you will by writing down one true thing and a bunch of nonsense things.

Show your work: If you use a result from the textbook that we haven't learned in class, tell me where you found it in the textbook. If you do partial fraction decomposition, show how you solved for your constants. This exam is not testing to see if you can get the right answers. It's testing to see if you know calculus.

Check your answers: Take the time before you turn in your test to make sure you have read the directions correctly and in their entirety, that all your work shown is correct, and that you have clearly stated your answer (by boxing or circling it where appropriate).

Pace yourself: If you're stuck on a problem, move on and come back to it later. Don't risk forcing yourself to give partial answers if you run out of time near the end of the test. Do the easy ones first. There are 80 points on this exam. That means you should budget about 0.6 minute(s) for each point a problem is worth in order to complete the exam in time.

Reminder: The only internet resource that you are allowed to use in conjunction with the exam are the notes that I have written and posted to Canvas. If you use any other online resources or if you communicate with other individuals during the exam, you will receive a zero on the exam and you will be reported to the university for academic misconduct.

1. **Vocabulary** (5 points each)

   (a) What is the **index variable** of the sequence $\{n_k\}_{k=0}^{\infty}$? A single-letter answer is acceptable here.

   k

   (b) Fill in the hypotheses of the **Monotone Convergence Theorem**: If a sequence $\{a_n\}_{n=0}^{\infty}$ is bounded and monotone for all $n \geqslant n_0$, then $\{a_n\}_{n=0}^{\infty}$ converges to some limit, L.

   (c) What is the 3rd **partial sum** (S_3) of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$? You don't have to simplify your answer.

   $S_3 = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} = \frac{8}{10} = \frac{4}{5}$

2. **True/False** (4 points each). For each statement, *circle* the word "true" if the statement is true and "false" otherwise. If you're writing on your own paper, write down the *entire* word "True" or "False." No work is necessary, but you can show work for some partial credit if you get the answer wrong.

   (a) **True** or **False**? The sequence defined by $a_1 = 4$ and $a_{n+1} = \frac{a_n}{a_{n-1}}$ is given in explicit form.

   (b) **True** or **False**? The series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{r-1}{3}\right)^n$ converges if $-2 < r < 4$.

   (c) **True** or **False**? The Monotone Convergence Theorem implies that the sequence defined by $c_n = \dfrac{\cos\left(\frac{n\pi}{6}\right)}{n}$ for $n \geqslant 0$ converges.

3. (4 points) Give an example of a sequence in explicit form which converges to -1.

$a_n = -1 + \frac{1}{n}$ for $n \geq 1$

4. (4 points) Give an example of a sequence in recursive form which diverges to ∞.

$a_0 = 0$

$a_{n+1} = a_n + 1$

5. (4 points each) For each of the following, give an example of a sequence OR state that no such sequence exists and give a brief explanation of why. You may write your answer in explicit or recursive form. If you can't come up with a formula, you may draw a graph for partial credit.

   (a) Give an example of a sequence which is *not* eventually monotone, but which converges to some limit (OR state that no such sequence exists and give a brief explanation why).

$a_n = \left(-\frac{1}{2}\right)^n$ for $n \geq 0$

   (b) Give an example of a sequence $\{b_n\}_{n=1}^{\infty}$ which is eventually increasing but with $\lim_{n\to\infty} b_n = -\infty$ (OR state that no such sequence exists and give a brief explanation why).

No such sequence exists since a sequence that eventually increases must be bounded from below.

   (c) Give an example of a sequence which is bounded, but does not converge (OR state that no such sequence exists and give a brief explanation why).

$a_n = (-1)^n$ for $n \geq 0$

6. (4 points each) Consider the sequence $\{a_n\}_{n=1}^{\infty} = \{\sin(2\pi), \sin(\frac{3\pi}{2}), \sin(\frac{4\pi}{3}), \sin(\frac{5\pi}{4}), ...\}$

   (a) Write this sequence in explicit form.

$$a_n = \sin\left(\frac{n+1}{n} \cdot \pi\right) \quad \text{for } n \geq 1$$

   (b) Compute $\lim_{n\to\infty} a_n$ (or state that the limit does not exist).

$$\lim_{n\to\infty} \sin\left(\frac{n+1}{n} \cdot \pi\right) = \sin\left(\lim_{n\to\infty} \frac{n+1}{n} \cdot \pi\right)$$

$$= \sin(\pi)$$

$$= 0$$

7. (8 points) Consider the sequence defined by $a_n = \frac{3n}{n+1}$ for $n \geqslant 0$. Find an explicit n_0 so that for all $n \geqslant n_0$, $|a_n - 3| < 10^{-100}$. Make sure to state your final answer as "$n_0 = \ldots$"

Want: $|a_n - 3| < 10^{-100}$

LHS: $|a_n - 3| = \left|\frac{3n}{n+1} - 3\right| = \left|\frac{3n - 3(n+1)}{n+1}\right|$

$= \left|\frac{-3}{n+1}\right| = \frac{3}{n+1}$

Want: $\frac{3}{n+1} < 10^{-100} \longleftrightarrow 3 \cdot 10^{100} < n+1$

$\longleftrightarrow 3 \cdot 10^{100} - 1 < n$

$n_0 = 3 \cdot 10^{100}$ works

8. (7 points) Write the number $7.\overline{31} = 7.3131313131...$ as a ratio of two integers (this is the one problem on the test where you have to "simplify" your answer).

$$7.\overline{31} = 7 + \frac{31}{100} + \frac{31}{10000} + \frac{31}{1000000} + \cdots$$

$$= 7 + \frac{31}{100}\left(1 + \frac{1}{100} + \frac{1}{10000} + \cdots\right)$$

$$= 7 + \frac{31}{100}\left(\frac{1}{1 - \frac{1}{100}}\right)$$

$$= 7 + \frac{31}{99}$$

$$= \frac{724}{99}$$

9. (10 points) Compute $\sum_{n=7}^{\infty} \frac{5}{(n+2)(n+3)}$. Show your work for all stages, including any partial fraction decomposition you might be doing.

$$\frac{5}{(n+2)(n+3)} = \frac{A}{n+2} + \frac{B}{n+3} \rightarrow 5 = A(n+3) + B(n+2)$$

$$n = -2 \rightarrow 5 = A$$

$$n = -3 \rightarrow 5 = -B \rightarrow B = -5$$

$$\sum_{n=7}^{\infty} \frac{5}{(n+2)(n+3)} = \sum_{n=7}^{\infty} \frac{5}{n+2} - \frac{5}{n+3}$$

$$S_k = \frac{5}{9} - \cancel{\frac{5}{10}} + \cancel{\frac{5}{10}} - \cancel{\frac{5}{11}} + \cdots + \cancel{\frac{5}{k+2}} - \frac{5}{k+3} = \frac{5}{9} - \frac{5}{k+3}$$

$$\sum_{n=7}^{\infty} \frac{5}{n+2} - \frac{5}{n+3} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{5}{9} - \frac{5}{k+3} = \boxed{\frac{5}{9}}$$